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Distributed Control and Simulation of a Bernoulli-Euler Beam

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The optimal distributed control of a Bernoulli-Euler beam is studied. A distributed parameter model of the system dynamics is derived, and this model is used to solve the linear quadratic regulator control problem. The exact model avoids the potentially destabilizing effects of control spillover, as it reproduces the beam dynamics at all frequencies, insofar as the mathematical model represents the actual physical structure. Functional Riccati equations are derived, and numerical procedures are developed to iteratively converge on a solution. The solutions represent gain surfaces, which relate the measured state variables to the control forces at arbitrary points along the beam. An efficient procedure for simulating the response of the closed-loop system to initial conditions and disturbance forces is then developed. Simulation results are presented that indicate that the control law successfully suppresses structural vibration. Issues concerning implementation of the distributed control system are addressed. Finally, an extension of the formulation to multiple beam systems, such as space frames, is discussed.

Introduction

THE control of flexible structures has been an ongoing field of research for the past few decades, due in some part to the advent of large aerospace structures. These structures tend to have relatively low modal frequencies. Moreover, these modes tend to be densely spaced. Quite often, the bandwidth of both disturbances acting on the structure and control forces imparted to it encompass many of these modes. This makes it difficult, for example, to slew a large spacecraft without exciting some vibrational modes. In situations where precise pointing accuracy is required, some method of controlling these vibrations must therefore be present. As a result, much effort has been directed toward extending proven methods of control theory to these types of plants.

Typically, a structure is discretized into a finite element model, and a corresponding state-space approximation is generated. Modern control techniques are applied to the state-space model, and control laws are developed that relate measurements at discrete points to actuation signals at those same or other points. Unfortunately, an extremely fine discretization is usually required to avoid such problems as control spillover, where the controller excites unmodeled modes, causing unwanted structural vibration. An alternative to a modal or lumped parameter approach is a distributed control model, where the control law is based on the partial differential equations that describe the structure. For truly distributed control, continuous measurements are made over the entire spatial domain of the system, and control forces are likewise applied. This paper presents the application of such a scheme to the control of a Bernoulli-Euler beam.

Distributed control theory is by no means a new field of research. Indeed, much of the essential groundwork was performed in the early 1960s.[1] However, until recently, the computing capabilities available for solving such problems had been inadequate. Researchers dealt primarily with extremely simple systems (such as one-dimensional diffusion equations), for which semianalytical results were available. With the advent of faster computing devices, solutions to moderately complex distributed control problems will soon be within reach. It is therefore necessary to develop and test algorithms that can be used to solve these problems. As a first step, this paper addresses the distributed control of a Bernoulli-Euler beam. This problem is sufficiently difficult so that an analytical feedback law is unavailable, yet simple enough to make apparent the salient features of the theory.

The finite time open-loop control problem for framelike structural systems with discrete actuators has been solved previously.[2,3] Distributed parameter models are used in the formulation, which result in control histories that take into account the high-frequency behavior of the structural elements without resorting to a highly discretized finite element model. However, this formulation does not carry over immediately into the closed-loop configuration. As a result, the closed-loop formulation developed here bears little resemblance to the open-loop case.

In this paper, a review of distributed control theory for one-dimensional systems is first presented. An extensive treatment of systems of artibrary dimension can be found in Refs. 4-6. This theory is then applied to the optimal control problem for the Bernoulli-Euler beam system, and the associated Riccati equations are derived. Numerical procedures for solving these equations are developed. The simulation of the open- and closed-loop systems is discussed, and an efficient simulation algorithm is developed. Implementation considerations, such as the effect of actuator/sensor delays, are then addressed.

Linear Quadratic Optimal Control Theory for One-Dimensional Systems

Distributed systems can be characterized in either of two forms. The first is an integral form, in which the response of the system at a particular time is determined by integrating (with respect to time and/or space) the product of the control input with a Green's function kernel. Here, the Green's function relates the response of the system at some arbitrary point and time to an impulse applied at some other point and time. Thus, this characterization is global in nature. Given this approach, it is possible to develop a distributed control theory.[7] The work of Brogan proceeds along these lines. However, the

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Green's function for an arbitrary system is extremely difficult to obtain. Indeed, analytical expressions are only available for the simplest of cases. The other characterization is differential in nature. Here, partial differential equations describe the local behavior of the system. This characterization is much easier to obtain, as the physical laws that describe the system are almost always local. Consequently, more effort has been spent on developing control theories based on this representation. Breakwell, for example, uses the differential description to obtain solutions to the boundary control of a simple flexible system.[8] The differential description of the distributed system will be used here throughout.

We will restrict our attention to one-dimensional, linear, time-invariant distributed systems. Such systems can be written in the form

$$\dot{\boldsymbol{x}}(x,t) = \boldsymbol{L}_x\boldsymbol{x}(x,t) + \boldsymbol{B}\boldsymbol{u}(x,t), \quad x \in [0,1], \quad t \in [0,\infty] \tag{1}$$

where $\boldsymbol{x}$ is a state vector exhibiting both spatial and temporal variation, $\boldsymbol{u}$ the distributed control input, $\boldsymbol{L}_x$ a matrix linear operator, and $\boldsymbol{B}$ the control influence matrix. Note that the spatial domain has been normalized to unity. In the subsequent development, both $\boldsymbol{L}_x$ and $\boldsymbol{B}$ are assumed to be constant. However, spatially varying system dynamics may also be modeled, with only minor changes in the development. In addition, the boundary conditions are assumed to be homogeneous and are expressed as

$$\boldsymbol{x}(0,t) = \boldsymbol{x}(1,t) = 0, \qquad t \in [0,\infty] \tag{2}$$

The development presented here applies to distributed control of systems with homogeneous boundary conditions only. It is therefore assumed that no controls are applied to the boundary of the system. Finally, the initial conditions are expressed as

$$\boldsymbol{x}(x,0) = \boldsymbol{x}_0(x), \qquad x \in [0,1] \tag{3}$$

The optimal regulator problem can be stated as follows. Given an arbitrary initial condition, determine the control required to return the system to the zero state while minimizing some cost criterion. We will assume a linear quadratic cost functional of the form

$$J = \frac{1}{2}\int_0^{\infty}\int_0^1 \left[\boldsymbol{x}(x,t)^T\boldsymbol{Q}\boldsymbol{x}(x,t) + \boldsymbol{u}(x,t)^T\boldsymbol{R}\boldsymbol{u}(x,t)\right] \mathrm{d}x\, \mathrm{d}t \tag{4}$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ are symmetric weighting matrices. This optimal control problem can be solved by extending the classical variational calculus approach to distributed systems, as described by Tzafestas and Nightingale.[9] Their results are summarized here for the one-dimensional case. The control law has the form

$$\boldsymbol{u}(x,t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\lambda(x,t) \tag{5}$$

where $\lambda(x,t)$ represents the distributed costate vector. The costate dynamics are given by

$$\dot{\lambda}(x,t) = -\boldsymbol{L}_x^*\lambda(x,t) - \boldsymbol{Q}\boldsymbol{x}(x,t) \tag{6}$$

where $\boldsymbol{L}_x^*$ is the adjoint of $\boldsymbol{L}_x$. Substituting Eq. (5) in Eq. (1) results in

$$\dot{\boldsymbol{x}}(x,t) = \boldsymbol{L}_x\boldsymbol{x}(x,t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\lambda(x,t) \tag{7}$$

Equations (6) and (7) represent the state-costate equations for the distributed control problem. It can be shown[5] that there exists a relation between the state and the costate of the form

$$\lambda(x,t) = \boldsymbol{P}_x\boldsymbol{x}(x,t) \tag{8}$$

where $\boldsymbol{P}_x$ is some linear matrix operator on $\boldsymbol{x}$. Substituting this form in Eqs. (6) and (7) results in a nonlinear matrix Riccati operator equation in $\boldsymbol{P}_x$. Such an equation is, in general, difficult to solve. Several approximate solution techniques are described in Juang and Dwyer,[10] Schaechter,[11] and Zambettakis et al.[12] However, it is possible to express the linear operator in a different form, so that a solution is attained easily by numerical methods. The assumed form of the solution is the same as used by Wang[4] and Tzafestas and Nightingale[9]

$$\lambda(x,t) = \int_0^1 \boldsymbol{S}(x,y)\boldsymbol{x}(y,t)\,\mathrm{d}y, \quad \boldsymbol{S}(0,y) = \boldsymbol{S}(1,y) = 0 \tag{9}$$

where $\boldsymbol{S}$ is the distributed-parameter analog of the Riccati matrix for lumped-parameter systems. This matrix is determined from the following functional Riccati equation:

$$\boldsymbol{L}_x^*\boldsymbol{S}(x,y) + \boldsymbol{S}(x,y)\boldsymbol{L}_y^{*T} + \boldsymbol{Q}\delta(x-y) - \int_0^1 \boldsymbol{S}(x,z)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S}(z,y)\,\mathrm{d}z = 0 \tag{10}$$

For complete generality, $\boldsymbol{S}$ must include generalized functions, such as Dirac delta functions and their derivatives, if necessary. It can be shown[4] that $\boldsymbol{S}$ is symmetric in its arguments [i.e., $\boldsymbol{S}(x,y) = \boldsymbol{S}(y,x)$]. The following boundary conditions must also be imposed:

$$\boldsymbol{S}(x,0) = \boldsymbol{S}(x,1) = 0 \tag{11}$$

Equation (10) is a functional nonlinear, matrix integro-partial-differential equation in x and y and represents the distributed parameter analog of the control algebraic Riccati equation. Note that we have assumed $\boldsymbol{S}$ to be time invariant, which corresponds to the steady-state linear quadratic regulator. For a finite time problem, the zero on the right side of Eq. (10) would be replaced by $-(\partial/\partial t)\boldsymbol{S}(x,y)$. Using the Riccati solution, the feedback law becomes

$$\boldsymbol{u}(x,t) = -\int_0^1 \boldsymbol{K}(x,y)\boldsymbol{x}(y,t)\,\mathrm{d}y, \quad \boldsymbol{K}(x,y) = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S}(x,y) \tag{12}$$

Optimal Control Applied to a Bernoulli-Euler Beam

In this section we will apply the distributed control formulation developed earlier to the equation of motion of a Bernoulli-Euler beam. We will assume the beam to be pinned at both ends, which ensures that the boundary conditions are homogeneous for the particular choice of state vector given here. A diagram of the physical system is shown in Fig. 1. In dimensional form, the beam dynamics are described by

$$EI\frac{\partial^4}{\partial x_d^4}w_d(x_d,t_d) + \rho A\frac{\partial^2}{\partial t_d^4}w_d(x_d,t_d) = f_d(x_d,t_d)$$
$$x_d \in [0,L], \qquad t_d \in [0,\infty] \tag{13}$$

where x_d and t_d are the dimensionalized spatial and temporal variables, respectively, w_d the transverse deflection, L the beam length, f_d the applied distributed force, EI the bending

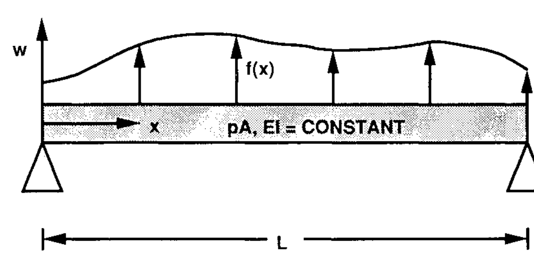

Fig. 1 Schematic of the Bernoulli-Euler beam system.

stiffness, and ρA the beam mass per unit length. To this equation we must add the boundary conditions

$$w_d(0,t_d)=\frac{\partial^2}{\partial x_d^2}w_d(0,t_d)=w_d(L,t_d)=\frac{\partial^2}{\partial x_d^2}w_d(L,t_d)=0$$

$$t_d\in[0,\infty] \tag{14}$$

and the initial conditions

$$w_d(x_d,0)=w_{0_d}(x_d),\qquad \frac{\partial}{\partial t_d}w_d(x_d,0)=\dot{w}_{0_d}(x_d) \tag{15}$$

Introducing nondimensional variables according to

$$x=\frac{x_d}{L},\qquad t=t_d\sqrt{\frac{EI}{\rho AL^4}} \tag{16}$$

and the nondimensional deflection and distributed force as

$$w(x,t)=\frac{1}{L}w_d(x_d,t_d),\qquad f(x,t)=\frac{L^3}{EI}f_d(x_d,t_d) \tag{17}$$

leads to the nondimensional form of the equation of motion

$$\frac{\partial^4}{\partial x^4}w(x,t)+\frac{\partial^2}{\partial t^2}w(x,t)=f(x,t)=f_c(x,t)+f_n(x,t) \tag{18}$$

Here, f_c and f_n represent the normalized distributed control and disturbance forces, respectively. The cost functional for this system will be expressed by

$$J_d=\frac{1}{2}\int_0^\infty\int_0^L\left\{q_U EI\left[\frac{\partial^2 w_d}{\partial x_d^2}\right]^2+q_T\rho A\left[\frac{\partial w_d}{\partial t_d}\right]^2+r\frac{L^4}{EI}f_d^2\right\}\mathrm{d}x_d\,\mathrm{d}t_d \tag{19}$$

Thus, q_U represents a weighting on deformational potential energy, q_T represents a weighting on kinetic energy, and r weighs control effort. The physical constants in the term involving f_d are introduced so that all three weights have the same units. The cost functional can then be normalized, yielding

$$J=\frac{1}{2}\int_0^\infty\int_0^1\left\{q_U\left[\frac{\partial^2 w}{\partial x^2}\right]^2+q_T\left[\frac{\partial w}{\partial t}\right]^2+rf^2\right\}\mathrm{d}x\,\mathrm{d}t \tag{20}$$

where the nondimensional cost is defined by

$$J=\frac{1}{\sqrt{EI\rho Al^2}}J_d \tag{21}$$

We must now cast the beam equation into the form given by Eq. (1). The well posedness of the control problem depends on the particular choice of state variables used in the state-space representation for Eq. (18).[13] For this problem, well posedness is guaranteed if we make the following associations

$$\boldsymbol{x}(x,t)=\begin{bmatrix}\dfrac{\partial^2}{\partial x^2}w(x,t)\\[2mm]\dfrac{\partial}{\partial t}w(x,t)\end{bmatrix},\qquad u=f_c(x,t) \tag{22}$$

The components of the state vector then represent normalized curvature and velocity along the beam. Equivalently, the components represent elastic potential and kinetic energy densities, respectively. The matrix operators are then

$$\boldsymbol{L}_x=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\frac{\partial^2}{\partial x^2},\qquad \boldsymbol{b}=\begin{bmatrix}0\\1\end{bmatrix} \tag{23}$$

The adjoint matrix operator follows on integrating by parts, leading to

$$\boldsymbol{L}_x^*=\boldsymbol{L}_x^T=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\frac{\partial^2}{\partial x^2} \tag{24}$$

The assumed form for the cost functional is given by

$$J=\frac{1}{2}\int_0^\infty\int_0^1\left[\boldsymbol{x}(x,t)^T\boldsymbol{Q}\boldsymbol{x}(x,t)+ru(x,t)^2\right]\mathrm{d}x\,\mathrm{d}t \tag{25}$$

where

$$\boldsymbol{Q}=\begin{bmatrix}q_U&0\\0&q_T\end{bmatrix} \tag{26}$$

Hence, we have assumed a spatially independent cost criterion with no cross-weighting between curvature and velocity.

The Riccati gain matrix $\boldsymbol{S}$ is given by

$$\boldsymbol{S}(x,y)=\boldsymbol{S}(y,x)=\begin{bmatrix}s_{11}(x,y)&s_{12}(x,y)\\s_{12}(x,y)&s_{22}(x,y)\end{bmatrix} \tag{27}$$

Because of the symmetry of $\boldsymbol{S}$, only three of the four Riccati equations are unique. One of these equations involves s_{11} and is not needed, since s_{11} will not appear in the control law. The remaining two integro-partial-differential equations are

$$\nabla^2 s_{12}(x,y)=q_U\delta(x-y)-\frac{1}{r}\int_0^1 s_{12}(x,z)s_{12}(z,y)\,\mathrm{d}z \tag{28}$$

$$\int_0^1 s_{22}(x,z)s_{22}(z,y)\,\mathrm{d}z=r(q_U+q_T)\delta(x-y)-\int_0^1 s_{12}(x,z)s_{12}(z,y)\,\mathrm{d}z \tag{29}$$

where ∇^2 is the Laplacian operator in two dimensions. It is a fortunate circumstance that s_{22} does not appear in Eq. (28). As a result, this equation can be solved independently, and its solution used to solve Eq. (29). The solutions thus obtained for s_{12} and s_{22} correspond to the curvature-to-force and velocity-to-force feedback kernels, respectively.

Numerical Solution of the Riccati Equations

Previous attempts to obtain a numerical solution to the optimal distributed control problem have most often dealt with the operator form of the Riccati equation.[14] Usually, the solution is expressed as a series expansion of spatial differential operators of increasing order.[10] In some cases, the distributed control law is only solved at points where discrete controls are to be applied, which leads to a slightly suboptimal design.[15] However, in this formulation, the functional form of the Riccati equations leads naturally to a numerical solution procedure. Because of the fundamental differences in the forms of Eqs. (28) and (29), a separate algorithm is developed for each equation.

Solution of the First Riccati Equation

Equation (28) is solved using finite differencing and a modified relaxation algorithm. We begin by discretizing the spatial variables according to

$$x_i=\frac{i}{N},\qquad i=0,\ldots,N \tag{30}$$

and defining the mesh points

$$s_{ij}=s_{12}(x_i,y_j) \tag{31}$$

A simple approximation to the Laplacian operator is then

$$\nabla^2 s_{12}(x_i,y_j)\approx N^2[\Delta_{ij}-4s_{ij}] \tag{32}$$

where

$$\Delta_{ij} = s_{i-1j} + s_{i+1j} + s_{ij-1} + s_{ij+1} \tag{33}$$

The forcing term in Eq. (28) can be approximated by

$$q_U\,\delta(x-y) \approx Nq_U\,\delta_{ij} \tag{34}$$

where δ_{ij} is the discrete Kronecker delta function. The integral term can be approximated by using the trapezoidal rule, leading to

$$\frac{1}{r}\int_0^1 s_{12}(x_i,z)s_{12}(z,y_j)\,\mathrm{d}z \approx \frac{1}{rN}\sum_{k=0}^{N} s_{ik}s_{kj} = \frac{1}{rN}\left[I_{ij} + \left(s_{ii}+s_{jj}\right)\left(1-\frac{1}{2}\delta_{ij}\right)\right] \tag{35}$$

where

$$I_{ij} = \sum_{\substack{k=0\\k\neq i\\k\neq j}}^{N} s_{ik}s_{kj} \tag{36}$$

Note that, in Eqs. (32) and (35), the terms involving s_{ij} have been isolated from terms involving neighboring points. Collecting the approximate expressions, we have, for the finite difference equation,

$$N^2\Delta^{\beta}_{ij} - 4s_{ij} = Nq_U\,\delta_{ij} - \frac{1}{rN}I_{ij} - \frac{1}{rN}\left(s_{ii}+s_{jj}\right)\left(1-\frac{1}{2}\delta_{ij}\right)s_{ij} \tag{37}$$

Thus, given an initial estimate for the solution at each mesh point s_{ij}^0, the entire mesh is successively iterated according to the rule

$$s_{ij}^{n+1} = s_{ij}^{n} - \omega e_{ij}^{n}, \qquad 0<\omega<2 \tag{38}$$

In Eq. (38), e_{ij}^n represents the residual error at each mesh point at the nth iteration. An expression for this error is obtained by solving Eq. (37) for s_{ij}, which yields

$$e_{ij} = s_{ij} - \frac{N^2\Delta_{ij} + (1/rN)I_{ij} - Nq_U\,\delta_{ij}}{4N^2 - (1/rN)(s_{ii}+s_{jj})\left[1-(1/2)\delta_{ij}\right]} \tag{39}$$

Also, ω is a relaxation parameter and can be adjusted to maximize the rate of convergence toward a solution.[16]

The condition for a converged solution is given by

$$|e_{ij}^n| < \epsilon, \qquad \forall i,j \tag{40}$$

where ϵ is some small positive constant. The relaxation method is guaranteed to converge when r approaches infinity [in this case, Eq. (28) reduces to Poisson's equation], and tests have shown that convergence is maintained over a wide range of values of q_U and r, provided ω is adjusted accordingly.

Solution of the Second Riccati Equation

Equation (29) does not have a well-behaved solution since it requires that the integral of $s_{22}(x,y)$ on the left side of the equation exactly cancels the delta function on the right. We therefore make the following substitution:

$$s_{22}(x,y) = \tilde{s}_{22}(x,y) + \sqrt{r(q_U+q_T)}\,\delta(x-y) \tag{41}$$

This is equivalent to identifying a collocated component in the velocity feedback kernel. Equation (29) then becomes

$$\int_0^1 \tilde{s}_{22}(x,z)\tilde{s}_{22}(z,y)\,\mathrm{d}z + 2\sqrt{r(q_U+q_T)}\,\tilde{s}_{22}(x,y) = -\int_0^1 s_{12}(x,z)s_{12}(z,y)\,\mathrm{d}z \tag{42}$$

Equation (42) can be solved directly, without relaxation iteration, as it does not contain any differential operators. If we first form two matrices corresponding to the mesh approximations to s_{12} and $\tilde{s}_{22}$,

$$S_{k2} = \left[s_{k2}(x_i,y_j)\right], \qquad S_{k2} = S_{k2}^T > 0, \qquad k = 1,2 \tag{43}$$

the integrals can then be represented by

$$\int_0^1 s_{k2}(x,z)s_{k2}(z,y)\,\mathrm{d}z \approx \frac{1}{N}S_{k2}^2, \qquad k = 1,2 \tag{44}$$

Equation (42) is then replaced by the nonlinear matrix equation

$$\frac{1}{N}\tilde{S}_{22}^2 + 2\sqrt{r(q_U+q_T)}\,\tilde{S}_{22} = \frac{1}{N}S_{12}^2 \tag{45}$$

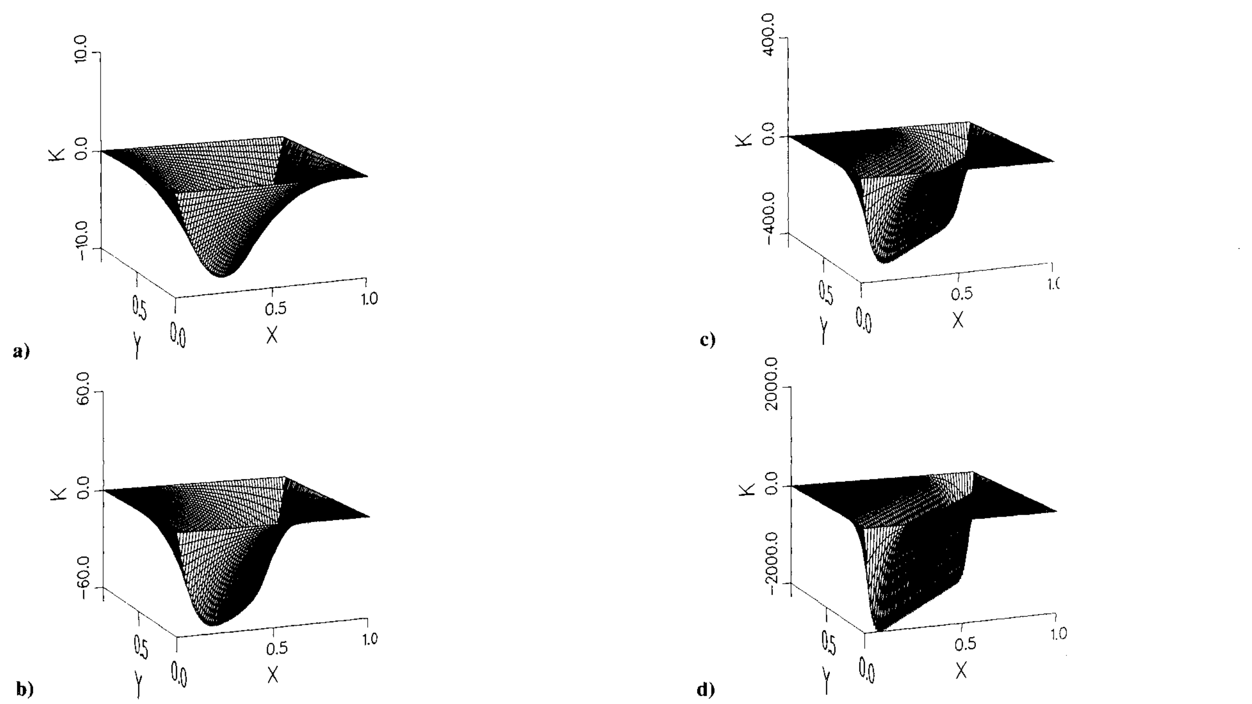

Fig. 2 Curvature-to-force feedback kernels for uniform beam: a) $r/q_U = 10^{-2}$; b) $r/q_U = 10^{-3}$; c) $r/q_U = 10^{-4}$; d) $r/q_U = 10^{-5}$.

This equation is solved by completing the square, which leads to

$$\left[\tilde{S}_{22} + N\sqrt{r(q_U+q_T)}\, I\right]^2 = -S_{12}^2 + N^2 r(q_U+q_T) I \tag{46}$$

The right side of Eq. (46) is symmetric and positive semidefinite. It therefore has the eigenvector decomposition

$$-S_{12}^2 + N^2 r(q_U+q_T) I = W\Lambda W^T \tag{47}$$

where Λ is a diagonal matrix with nonnegative entries. Finally, substituting Eq. (47) in Eq. (46) and solving for $\tilde{S}_{22}$ gives

$$\tilde{S}_{22} = W\Lambda^{1/2} W^T - N\sqrt{r(q_U+q_T)}\, I \tag{48}$$

Results of Solution Algorithms

The algorithms presented in the preceding subsections were implemented and tested on a digital computer. For a given set of parameters (q_U, q_T, and r), the gain surfaces were calculated using several different mesh refinements in order to demonstrate that the algorithms converged upon unique solutions. For all parameters of interest, this was found to be the case. For the first Riccati equation, the algorithm performed best when ω was set to 1.3.

The feedback control law for the beam contains both collocated and distributed components, due to the delta function in the expression for s_{22}. It has the form

$$f_c(x,t) = -\int_0^1 \left[k_1(x,y) \frac{\partial^2}{\partial y^2} w(y,t) + \tilde{k}_2(x,y) \frac{\partial}{\partial t} w(y,t) \right] \mathrm{d}y - \sqrt{\frac{q_U+q_T}{r}} \frac{\partial}{\partial t} w(x,t) \tag{49}$$

where

$$\begin{bmatrix} k_1(x,y) \\ \tilde{k}_2(x,y) \end{bmatrix} = \frac{1}{r} \begin{bmatrix} s_{12}(x,y) \\ \tilde{s}_{22}(x,y) \end{bmatrix} \tag{50}$$

Figure 2 shows curvature-to-force k_1 gain kernels for different weighting parameters, computed according to the algorithm presented earlier. It is clear that, as r decreases (i.e., more control authority is allowed), the control law becomes more and more localized and the magnitude of the gain becomes correspondingly larger. This makes sense physically. With more control authority, it is possible to suppress vibrational energy before it propagates very far. Consequently, one would expect the control force to be applied near the source of the disturbance. It is also interesting to observe that these control gain surfaces bear a close resemblance to those generated by Schaechter,[11] where he considered the problem of distributed estimation of the state of a stretched vibrating string. Finally, Fig. 3 presents typical velocity-to-force $\tilde{k}_2$ gain kernels.

Closed-Loop Simulation of the Controlled Beam

Given the control law, there remains the problem of actually simulating the response of the system. Various methodologies exist to achieve this end. At one extreme, the closed-loop equation is discretized in both space and time and then integrated forward in time. This constitutes a partial differential equation with mixed boundary and initial conditions. Although this method is widely used, it requires rather fine discretizations in both dimensions to achieve accurate results, and errors tend to accumulate in time. At the other extreme, one can transform the equation into the s-domain and search for analytic solutions. However, due to the distributed nature of the control law, this transformation results in an integro-partial-differential equation rather than a simple ordinary differential equation (as would be the case for the uncontrolled equation of motion). Because of the generality of the feedback gains, a general analytical solution is beyond reach.

In order to achieve accurate solutions with relatively coarse discretizations, a third alternative is proposed. The closed-loop equation is transformed, resulting in the just mentioned integro-partial-differential equation. At each desired complex frequency, a finite differencing scheme is used to solve for the displacement field. The data from a set of frequencies is collected, and a numerically robust inverse Laplace transform algorithm is used to convert the data back into the time domain.[17] Because the transformed equation is a boundary value problem, it is anticipated that its approximate solution will be more stable and accurate than the corresponding solution to the mixed problem associated with time-domain integration.

Thus, we first transform Eq. (18) into the frequency domain

$$\frac{\partial^4}{\partial x^4} \bar{w}(x,s) + s^2\left[\bar{w}(x,s) - s w_0(x) - \dot{w}_0(x)\right] = \bar{f}_c(x,s) + \bar{f}_n(x,s) \tag{51}$$

The normalized frequency s is related to the dimensional frequency s_d by

$$s = s_d \sqrt{\frac{\rho A l^4}{EI}} \tag{52}$$

In this way, we can relate the transform pair $w(x,t_d) \sim \bar{w}(x,s_d)$ with the pair $w(x,t) \sim \bar{w}(x,s)$. Substituting the transformed control law given by Eq. (49) leads to

$$\frac{\partial^4}{\partial x^4} \bar{w}(x,s) + \left[s^2 + \sqrt{\frac{q_U+q_T}{r}}\, s \right] \bar{w}(x,s) + \int_0^1 \left[k_1(x,y) \frac{\partial^2}{\partial y^2} \bar{w}(y,s) + s\tilde{k}_2(x,y) \bar{w}(y,s) \right] \mathrm{d}y = \bar{f}_n(x,s) + \dot{w}_0(x) + \left[s + \sqrt{\frac{q_U+q_T}{r}} \right] w_0(x) + \int_0^1 \tilde{k}_2(x,y) w_0(y)\, \mathrm{d}y \tag{53}$$

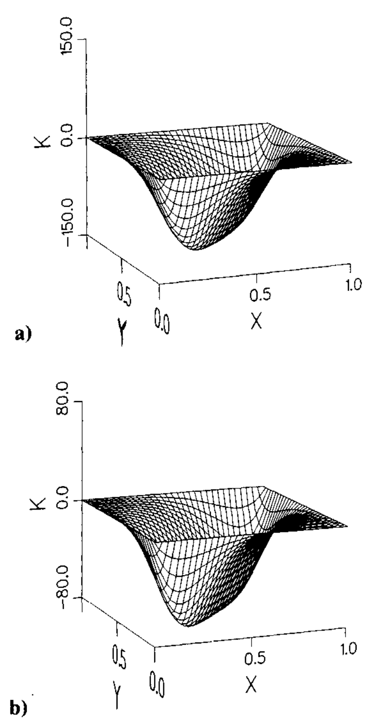

Fig. 3 Velocity-to-force feedback kernels for uniform beam (distributed component only): a) $r/q_U = 10^{-4}$, $q_T = 0$, b) $r/q_U = 10^{-4}$, $q_T = q_U$.

The term involving k_1 in this equation can be integrated by parts twice so that the derivative with respect to y operates on k_1. The boundary terms arising from this operation vanish due to the homogeneous boundary conditions. By making the following associations

$$k(x,y,s) = \frac{\partial^2}{\partial y^2} k_1(x,y) + s\bar{k}_2(x,y) \tag{54}$$

$$\bar{f}(x,s) = \bar{f}_n(x,s) + \dot{w}_0(x) + \left[s + \sqrt{\frac{q_U + q_T}{r}} \right] w_0(x) + \int_0^1 \bar{k}_2(x,y) w_0(y)\, \mathrm{d}y \tag{55}$$

the closed-loop equation reduces to

$$\frac{\partial^4}{\partial x^4} \bar{w}(x,s) + \left[s^2 + \sqrt{\frac{q_U + q_T}{r}}\, s \right] \bar{w}(x,s) + \int_0^1 k(x,y,s) \bar{w}(y,s)\, \mathrm{d}y = \bar{f}(x,s) \tag{56}$$

This equation must be solved numerically for each value of s needed to construct the time-domain response. To do so requires a spatial discretization of the domain. By defining the following terms

$$\bar{w}(s) \approx \{\bar{w}(x_i,s)\}, \quad \bar{f}(s) \approx \{\bar{f}(x_i,s)\}, \quad K(s) \approx [k(x_i,y_j,s)] \tag{57}$$

an approximation to Eq. (56) is obtained easily. The first term is replaced by the finite difference approximation

$$\frac{\partial^4}{\partial x^4} \bar{w}(x,s) \approx N^4 D^2 \bar{w}(s) \tag{58}$$

where

$$D = \begin{bmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & 0 \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ 0 & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{bmatrix} \tag{59}$$

The second term in Eq. (56) is trivially approximated by

$$\left[s^2 + \sqrt{\frac{q_U + q_T}{r}}\, s \right] \bar{w}(x,s) \approx \left[s^2 + \sqrt{\frac{q_U + q_T}{r}}\, s \right] I\bar{w}(s) \tag{60}$$

Finally, the third term is approximated using the trapezoidal rule:

$$\int_0^1 k(x,y,s) \bar{w}(y,s)\, \mathrm{d}y \approx \frac{1}{N} K(s) \bar{w}(s) \tag{61}$$

Collecting terms, the discretized equation becomes

$$\left[N^4 D^2 + \left(s^2 + \sqrt{\frac{q_U + q_T}{r}}\, s \right) I + \frac{1}{N} K(s) H \right] \bar{w}(s) = \bar{f}(s) \tag{62}$$

Thus, a single matrix inversion is required at each frequency. If the frequencies are given by

$$s = s_1, \ldots, s_n \tag{63}$$

then the solutions of Eq. (62) can be grouped according to

$$\bar{W} = \{\bar{w}(s_1) \ldots \bar{w}(s_n)\} \tag{64}$$

Finally, if the time-domain responses are represented by

$$W = \{w(t_1) \ldots w(t_n)\} \tag{65}$$

the inverse Laplace transform operation can be expressed as

$$W = \bar{W}M \tag{66}$$

By choosing appropriate values of s, the structure of M is such that a fast Fourier transform algorithm can be used. This decreases the computation time of the inverse transform operation significantly. Details on the values of s to be used, the structure of M, and the transform algorithm, in general, can be found in Ref. 17.

The Laplace transform-based algorithm was used to simulate the beam system under both open- and closed-loop configurations. Two types of disturbances were studied. The first was a nonzero initial condition with no distributed forcing. The beam was initially deformed into its second mode shape, $w_0 = \sin(2\pi x)$, and then released, as shown in Fig. 4. The controlled responses for various control and state penalties are shown in the figure. The damping effect of the control is obvious. The second disturbance consisted of a single impulse applied to the center of the beam, with the initial conditions set to zero. The results, shown in Figs. 5 and 6, indicate that the vibrations have been significantly suppressed before the disturbance reaches the boundaries of the system.

In order to test the robustness of the closed-loop system, time delays between sensing and actuation were simulated. This was accomplished in the Laplace domain by multiplying

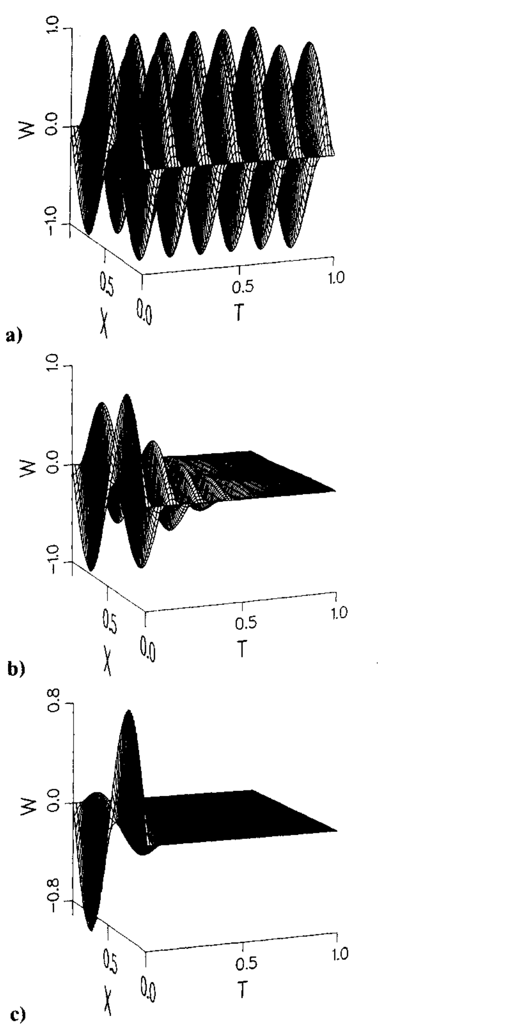

Fig. 4 Simulation of beam with sinusoidal initial condition: a) open loop; b) $r/q_U = 10^{-3}$, $q_T = q_U$; c) $r/q_U = 10^{-4}$, $q_T = q_U$.

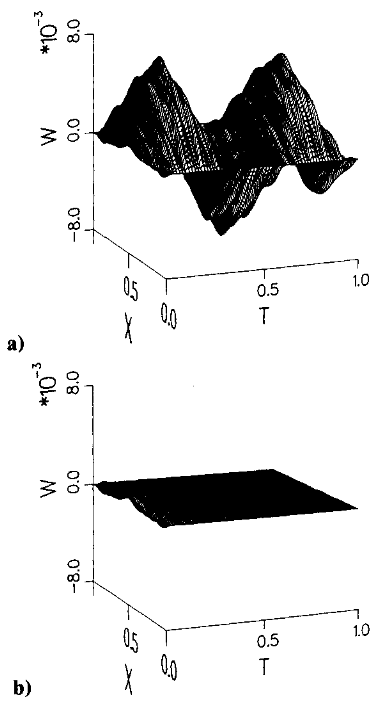

Fig. 5 Simulation of beam with impulsive disturbance (long time scale): a) open loop; b) $r/q_U = 10^{-4}$, $q_T = q_U$.

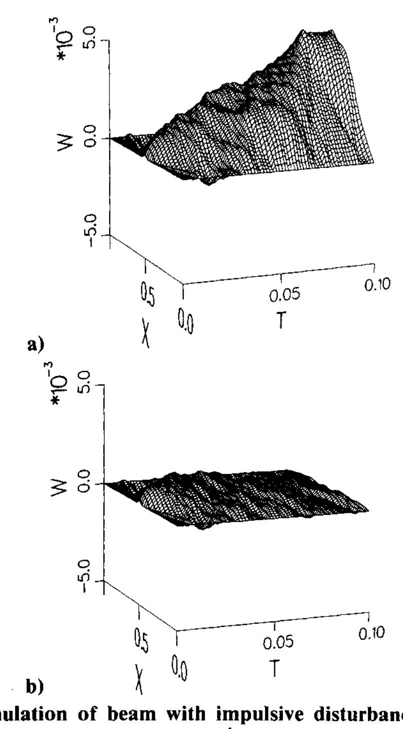

Fig. 6 Simulation of beam with impulsive disturbance (short time scale): a) open loop; b) $r/q_U = 10^{-4}$, $q_T = q_U$.

the control force by the exponential of the complex frequency scaled by a normalized delay time. The results of these simulations are shown in Fig. 7 for the sinusoidal initial condition and Fig. 8 for the impulsive disturbance. The potentially destabilizing effect of delay in the control loop is apparent. These results demonstrate the capability of the transform-based simulation approach to handle nonideal situations, such as actuator/sensor delays.

Implementation Considerations

Any physical control system is by necessity discrete in nature. Although it is, in some cases, possible to obtain a continuously distributed measurement of a quantity in some integrated sense (e.g., a piezoelectric film produces a voltage proportional to the integral of the strain over its area), applied forces and moments are always effected by actuators placed at discrete locations. With this in mind, it would appear as if a lumped parameter representation of the system would be an adequate model from which to derive spatially discretized control laws. However, the discretization associated with such an approach is usually performed ad hoc, without prior knowledge of the level of discretization required to achieve good closed-loop performance. On the other hand, a control design based on the distributed parameter model provides a good indication of the density and number of lumped actuators and/or sensors necessary to reproduce the distributed effect of the control law.

The distributed control laws developed in the preceding sections guarantee closed-loop stability of the system, provided that the physical system is described exactly by the Bernoulli-Euler beam equation. In reality, however, this model breaks down for high-frequency behavior predictions. For example, it predicts unbounded disturbance propagation velocity with increasing frequency, which is physically impossible. A more accurate mathematical representation of this system would be a Timoshenko or other higher-order beam model. To date, no distributed control solutions exist for such models. It is predicted, however, that higher-order solutions will bear the same general resemblance to the Bernoulli-Euler solution.

An extension of this formulation to multiple-beam structures (e.g., space frames and trusses) is currently being addressed. In this case, extensional and torsional vibrations must also be included in each beam model. Consequently, the state vector for each beam will have eight elements, rather than two. However, the system can still be described in the form given by Eq. (1). This is accomplished by associating a local coordinate system with each structural element and grouping the state vectors for each element into a global state vector. The linear operation L_x and the matrix $\boldsymbol{B}$ are then block diagonal. However, the boundary operators become considerably more complex, relating individual elements of the global state vector at

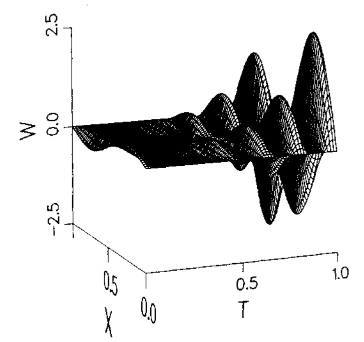

Fig. 7 Closed-loop simulation of beam with sinusoidal initial condition and normalized control delay time of 0.075: $r/q_U = 10^{-4}$, $q_T = q_U$.

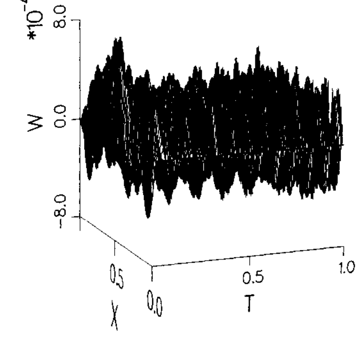

Fig. 8 Closed-loop simulation of beam with impulsive disturbance and normalized control delay time of 0.075: $r/q_U = 10^{-4}$, $q_T = q_U$.

the boundaries. It is therefore not immediately obvious how to guarantee the well posedness of a particular representation. Furthermore, if control forces are to be applied at structural junctions (as is often the case), the boundary conditions become nonhomogeneous and the integration by parts used to derive Eq. (10) must be preformed more rigorously. Finally, for multiple-beam systems, the decoupling observed between Eqs. (28) and (29) will most likely not occur.

Conclusions

The ability to compute and simulate distributed control laws has been demonstrated. The numerical algorithms developed to solve the associated Riccati equations were implemented and the gain surfaces calculated. The curvature-to-force feedback was determined to be purely distributed, whereas the velocity-to-force feedback consisted of both collocated and distributed components. Closed-loop simulations of the beam system indicated that the control law succeeded in suppressing unwanted structural vibration. Finally, the presence of sensing and/or actuation delays was shown to have a potentially destabilizing effect. The treatment of higher-order beam models and multiple-element structures is considerably more difficult and the corresponding control solutions are currently unavailable.

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